

SOME CLASSES OF PRERADICALS INDUCED BY RELATIVE INJECTIVITY

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To the memory of Francisco Raggi

ABSTRACT. In this work is presented a weakened version of the left exact preradicals: the prehereditary preradicals. Also is presented a weakened version of the idempotent preradicals: the essentially idempotent preradicals. These new classes of preradicals are studied under the lattice theory. Each class serves to generalize the theorems of the injectivity respect to a torsion theory.

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1. INTRODUCTION

In this paper are introduced new classes of preradicals, the essentially idempotent preradicals, the prehereditary preradicals, and the autocostable preradicals. It is seen their relation with injectivity respect a preradical. First it is introduced the class of essentially idempotent preradicals as a generalization of the idempotent preradicals, as such, it is to hope that the essentially idempotent preradicals preserve some properties that the idempotent preradicals have, and so they do, for example, the class of essentially idempotent preradicals is closed under supremum as the the class of idempotent preradicals and their pretorsion free classes are closed under extensions.

The class of the prehereditary preradicals is introduced in this article as a generalization of the left exact preradicals or hereditary preradicals. It is proved that there is a correspondence between the linear filters and the prehereditary preradicals, this correspondence is not a lattice isomorphism but it is an monomorphism from the class of linear filters to the prehereditary preradicals. The preradicals give a perfect context to define essentiality with respect to a preradical with almost all the properties of the usual essentiality. It is studied the concept of purity to respect a preradical and are obtained almost all results

that are gotten when the preradical is asked to be a left exact radical. It is seen that with an idempotent radical are obtained all the usual properties.

It is studied the concept of injectivity respect to a preradical. It is obtained that for have almost all results is sufficient to ask that the preradical to be an idempotent radical. But if it is looked for a criterion alike the Baer criterion, it is obtained when the preradical is prehereditary. Also is always possible to define the injective hull relative to preradical. When the preradical is an idempotent radical it is unique with respect certain properties. Also is defined the pseudocomplemented submodules respect to a preradical, which give conditions to determine when the classes of injectives of two preradicals are the same. Then it is studied the torsion free injective modules that are called absolutely pure modules. To continue with the next class of preradicals defined the autocostable preradicals which are those whose pretorsion free classes are closed under relative injective hulls. It is seen that under certain requirements this implies the costability. Finally it is defined an assignment for any left module respect to a preradical. When the preradical is left exact radical it results the localization functor. Certain properties of this assignment give information of the preradical even if the preradical is not a left exact radical.

2. PRELIMINARIES

In this paper R always will denote an associative ring with 1, and by $R\text{-Mod}$ it will understand the category of unitary left modules over R , an excelent reference of the category of left R -modules is (10). Let M be an R -module and N a submodule of M . If N is essential in M this is denoted by $N \leq M$. The injective hull of M is denoted by $E(M)$.

A preradical σ over $R\text{-Mod}$ is a subfunctor of the identity, references for preradicals are (1) and (9). The class of the preradicals is ordered punctually, That is, if σ and τ are preradicals over $R\text{-Mod}$, $\sigma \leq \tau$ if $\sigma(M) \leq \tau(M)$ for any left R -module M . With this order the class of preradicals becomes a big complete lattice. It also, has to operations the product and the coproduct. If a preradical is idempotent under de product it is called an idempotent preradical. If is idempotent under de coproduct it is called a radical. The class of all idempotent preradicals over $R\text{-Mod}$ will be denote by $R\text{-id}$. The class of all radicals over $R\text{-Mod}$ will be denote by $R\text{-rad}$. A class of left R -modules is called a pretorsion class if it is closed under coproducts and quotients. A class of left R -modules is called a pretorsion free class if it is closed under products and submodules. If σ is a preradical the class $\mathbb{T}_\sigma = \{M \mid \sigma(M) = M\}$ is a pretorsion class. If \mathbb{T} is a pretorsion class then the assignation $\sigma_{\mathbb{T}}(M) = \sum\{N \leq M \mid N \in \mathbb{T}\}$ for any left R -module M is an idempotent preradical, this is a bijective correspondence between the pretorsion classes and idempotent preradicals. In the same way, if σ is preradical then $\mathbb{F}_\sigma = \{M \mid \sigma(M) = 0\}$ is a pretorsion free class. If \mathbb{F} is pretorsion free class then assignation $\sigma^{\mathbb{F}}(M) = \bigcap\{N \leq M \mid \sigma(M/N) = 0\}$ for any left R -module M is a radical, this is a bijective correspondence between the pretorsion free classes and the radicals. The elements of \mathbb{T}_σ are called σ -torsion modules. The elements of \mathbb{F}_σ are called σ -torsion free modules. A class of modules \mathcal{C} is closed under extensions if for

any short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ with $M', M'' \in \mathcal{C}$ then $M \in \mathcal{C}$. A pretorsion class which is closed under extensions is called a torsion class. A pretorsion free class which is closed under extensions is called a torsion free class. If σ is an idempotent preradical then \mathbb{T}_σ is a torsion class. If σ is a radical then \mathbb{F}_σ is a torsion free class.

The class of idempotent preradicals is closed under supremum, so it is possible for any preradical σ to obtain the greatest idempotent preradical below σ . It results to be $\sigma_{\mathbb{T}_\sigma}$. It is usually denoted by $\widehat{\sigma}$. Also, the class of radicals is closed under infimum, so for any preradical σ there is the least radical above σ . It results to be $\sigma^{\mathbb{F}_\sigma}$. It is usually denoted by $\bar{\sigma}$.

Let M be a left R -module and let N be a submodule of M . N is called a fully invariant submodule of M if $f(N) \subseteq N$ for any endomorphism f of M . Let σ be a preradical. Then $\sigma(M)$ is a fully invariant submodule of M . Moreover, N is a fully invariant submodule of M if and only if there is a preradical σ such that $\sigma(M) = N$. If N is a fully invariant submodule of M , it is defined:

$$\alpha_N^M(K) = \sum \{f(N) \mid f : M \longrightarrow K\}$$

$$\omega_N^M(K) = \bigcap \{f^{-1}(N) \mid f : K \longrightarrow M\}$$

for any left R -module K . It is easy to see that $\alpha_N^M(M) = N$ and $\omega_N^M(M) = N$. Moreover, for any preradical σ , $\sigma(M) = N$ if and only if $\alpha_N^M \leq \sigma \leq \omega_N^M$. For N a fully invariant submodule of M is defined $\widehat{N} := \widehat{\alpha_N^M}(M)$. Then α_N^M is an idempotent preradical if and only if $N = \widehat{N}$. In the same way, it is defined $\bar{N} = \bar{\omega_N^M}(M)$. Also, ω_N^M is a radical if and only if $N = \bar{N}$. Let S be a simple left R -module. Then S is injective if and only if $\alpha_S^{E(S)}$ is idempotent. The standard references for lattice aspects of preradicals and the alphas and omegas preradicals are the papers (6), (7) and (8).

A class of modules is called hereditary if it is closed under submodules. A preradical σ is called hereditary if it is idempotent and \mathbb{T}_σ is hereditary. A preradical is hereditary if and only if it is left exact. The class of all left exact preradicals over $R\text{-Mod}$ will be denoted by $R\text{-lep}$. This class is closed under infimum, so for any preradical σ there is the least left exact preradical $\tilde{\sigma}$ above σ . It is usually denoted by $\tilde{\sigma}$. It is easy to describe, $\tilde{\sigma}(M) = \sigma(E(M)) \cap M$ for any left R -module M . A set of left ideals \mathbb{I} that satisfies:

- If $I \in \mathbb{I}$ and $I \subseteq J \leq R$ then $J \in \mathbb{I}$.
- If $I, J \in \mathbb{I}$ then $I \cap J \in \mathbb{I}$.
- If $a \in R$ and $I \in \mathbb{I}$ then $(I : a) \in \mathbb{I}$.

is called a left linear filter. If σ is a left exact preradical then it is defined $\mathbb{I}_\sigma = \{I \leq R \mid \sigma(R/I) = R/I\}$. For a linear filter, it is defined $\sigma_{\mathbb{I}}(M) = \{x \in M \mid \text{ann}(x) \in \mathbb{I}\}$, this assignation is a bijective correspondence between the left exact preradicals and the left linear filters. A left linear filter \mathbb{I} is called a left Gabriel filter if it satisfies: If $I \in \mathbb{I}$ and $J \leq R$ is such that for any $a \in I$ ($J : a$) $\in \mathbb{I}$ then $J \in \mathbb{I}$. The previous correspondence induces a bijective correspondence between the left exact radicals and the left Gabriel filters.

Let E and E' be injective left R -modules, it is said that E and E' are related if there is an imbedding of E in a product of copies of E' and there is an embedding of E' in a product of copies of E . It is easy to see that this is an equivalence relation. A class of equivalence is called an hereditary torsion theory, a good reference is (5). There is a bijective correspondence between the hereditary torsion theories and the left exact radicals.

3. ESSENTIALLY IDEMPOTENT PRERADICALS

Let σ be a preradical over $R\text{-Mod}$. Then σ is called essentially idempotent if $\sigma(M) \neq 0$ implies $\widehat{\sigma}(M) \neq 0$ for any left R -module M . It is observed that any idempotent preradical is essentially idempotent, so the property of being essentially idempotent is a generalization of being idempotent. The class of all the essentially idempotent preradicals over $R\text{-Mod}$ is denoted by $R\text{-eid}$. The last remark could be restated as $R\text{-id} \subseteq R\text{-eid}$. From this last fact it could happen that $R\text{-eid}$ is not a set, since $R\text{-id}$ is not always a set. As the supremum of a family of idempotent preradicals is idempotent, it is expected that the same happens for essentially idempotent preradicals.

Proposition 1. *Let $\{\sigma_i\}_{i \in I}$ be a family of essentially idempotent preradicals over $R\text{-Mod}$. Then $\bigvee_{i \in I} \sigma_i$ is an essentially idempotent preradical.*

Proof. Let M be a left R -module with $(\bigvee_{i \in I} \sigma_i)(M) \neq 0$, then there is $i \in I$ with $\sigma_i(M) \neq 0$ and by hypothesis it follows that $\widehat{\sigma}_i(M) \neq 0$. So $(\bigvee_{i \in I} \widehat{\sigma}_i)(M) \neq 0$ and since $\bigvee_{i \in I} \widehat{\sigma}_i \leq \widehat{\bigvee_{i \in I} \sigma_i}$. Therefore $\widehat{\bigvee_{i \in I} \sigma_i}(M) \neq 0$ as desired. ■

By the last proposition, for any preradical σ over $R\text{-Mod}$, it is possible to construct the greatest essentially idempotent preradical below σ . It will be denoted by σ° . As expected, σ° is the supremum of all essentially idempotent preradicals below σ . By the previous proposition σ° is an essentially idempotent preradical. It is observed that a preradical σ is essentially idempotent if and only if $\sigma^\circ = \sigma$. Also it is important to remember that the class $R\text{-id}$ is not closed under infimum, even finite ones. This pathology is preserved by the class $R\text{-eid}$, it is considered the next example. Let R be the ring of the integers, let σ be the socle and let τ be the divisible part. As σ and τ are idempotents they are essentially idempotents, but $(\sigma \wedge \tau)(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_p$ for any prime p . Then $(\sigma \wedge \tau)^2(\mathbb{Z}_{p^\infty}) = 0$ which implies that $(\widehat{\sigma \wedge \tau})(\mathbb{Z}_{p^\infty}) = 0$, so $\sigma \wedge \tau$ is not essentially idempotent. From here it is seen that the

infimum of idempotent preradicals is not essentially idempotent. Which also implies that the infimum of essentially idempotent preradicals is not essentially idempotent. The last said that $R\text{-eid}$ is not a sublattice of $R\text{-pr}$ and $R\text{-id}$ is not a sublattice of $R\text{-eid}$.

As it is expected, $R\text{-eid}$ has a natural way to be described as a complete lattice. That is for any family $\{\sigma_i\}_{i \in I}$ of essentially idempotent preradicals the supremum is the usual supremum in $R\text{-pr}$, but the infimum results $(\bigwedge_{i \in I} \sigma_i)^\circ$. The next proposition tells that the operator $^\circ$ over $R\text{-pr}$ is an interior operator.

Proposition 2. *The assignation $^\circ : R\text{-pr} \longrightarrow R\text{-pr}$ given by $\sigma \mapsto \sigma^\circ$ for any preradical over $R\text{-Mod}$ σ is a monotone, deflatory and idempotent operator over $R\text{-pr}$.*

From the fact that $R\text{-id} \subseteq R\text{-eid}$, it follows that $\widehat{\sigma}(M) \leq \sigma^\circ(M)$ for any left R -module M .

Remark 1. *Let σ be a preradical over $R\text{-Mod}$. Then:*

- (1) *If σ is an essentially idempotent preradical over $R\text{-Mod}$ and M is a left R -module with $\sigma(M) = M$, then $\sigma^\circ(M) = M$.*
- (2) $\mathbb{T}_\sigma = \mathbb{T}_{\sigma^\circ} = \mathbb{T}_{\widehat{\sigma}}$.
- (3) $\widehat{\sigma^\circ} = \widehat{\sigma}$.

It is well known that for any idempotent preradical σ , the associated radical $\bar{\sigma}$ is an idempotent radical. The next result has the same spirit that this one.

Proposition 3. *Let σ be an essentially idempotent preradical over $R\text{-Mod}$. Then $\bar{\sigma}$ is an essentially idempotent radical.*

Proof. Let M be a left R -module with $\bar{\sigma}(M) \neq 0$, then $\sigma(M) \neq 0$. It follows that $\widehat{\sigma}(M) \neq 0$ and as $\widehat{\sigma}(M) \leq (\bar{\sigma})(M)$ the desired result is obtained. ■

Proposition 4. *Let S be a simple left R -module. If $\alpha_S^{E(S)}$ is essentially idempotent then $\alpha_S^{E(S)}$ is idempotent.*

Proof. First it is remembered that $\alpha_S^{E(S)}$ is an atom in the lattice $R\text{-pr}$. By this $\widehat{\alpha_S^{E(S)}}$ has two options to be $\alpha_S^{E(S)}$ or to be 0. But $\alpha_S^{E(S)}(E(S)) = S \neq 0$ which means that $\widehat{\alpha_S^{E(S)}}(E(S)) \neq 0$. So $\widehat{\alpha_S^{E(S)}} = \alpha_S^{E(S)}$. ■

Corollary 1. *Let R be a ring. Then R is a V-ring if and only if every atom in R -pr is essentially idempotent.*

Proposition 5. *Let $\{\sigma_i\}_{i \in I}$ be a family of preradicals over R -Mod. Then $\widehat{\bigwedge_{i \in I} \sigma_i} = \widehat{\bigwedge_{i \in I} \hat{\sigma}_i}$.*

Proof. It is observed that $\mathbb{T}_{\bigwedge_{i \in I} \sigma_i} = \bigcap_{i \in I} \mathbb{T}_{\sigma_i} = \bigcap_{i \in I} \mathbb{T}_{\hat{\sigma}_i} = \mathbb{T}_{\bigwedge_{i \in I} \hat{\sigma}_i}$. ■

Corollary 2. *Let $\{\sigma_i\}_{i \in I}$ be a family of preradicals over R -Mod such that $\bigwedge_{i \in I} \sigma_i$ is essentially idempotent. Then $\bigwedge_{i \in I} \hat{\sigma}_i$ is essentially idempotent.*

Remark 2. *Let σ be a preradical over R -Mod. Then σ is an essentially idempotent preradical if and only if $\mathbb{F}_{\hat{\sigma}} = \mathbb{F}_{\sigma}$.*

From the classical theory of preradical it is a well known fact that when σ is an idempotent preradical, \mathbb{F}_{σ} is closed under extensions. The next remark generalizes the previous fact.

Remark 3. *Let σ be an essentially idempotent preradical over R -Mod. Then \mathbb{F}_{σ} is closed under extensions.*

Proposition 6. *Let σ be an essentially idempotent radical over R -Mod. Then σ is an idempotent radical.*

Proof. Let M be a left R -module. Since σ is a radical, also σ^2 is radical, then $\sigma^2(M/\sigma^2(M)) = 0$. This implies that $\sigma(M/\sigma^2(M)) = 0$, but $\sigma(M/\sigma^2(M)) = (\sigma^2 : \sigma)(M)/\sigma^2(M)$. From here it is obtained that $\sigma(M) \leq (\sigma^2 : \sigma)(M) = \sigma^2(M)$ and the desired result is followed. ■

It is known that for any left R -module M and any fully invariant submodule N , the preradical α_N^M is idempotent if and only if $\hat{N} = N$. In the same spirit it is stated the next remark.

Remark 4. *Let M be a left R -module and N a no zero fully invariant submodule of M . If α_N^M is an essentially idempotent preradical then $\hat{N} \neq 0$.*

It is considered the ring $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ and the ideal $I = \mathbb{Z}_4 \times 2\mathbb{Z}_4$. It is observed that $\alpha_I^R(0 \times \mathbb{Z}_4) = 0 \times 2\mathbb{Z}_4$ and $\widehat{\alpha_I^R}(0 \times \mathbb{Z}_4) = 0$ which means that α_I^R is not an essentially idempotent preradical. Also that $\widehat{I} = \mathbb{Z}_4 \times 0$. This tells that in general M is not a test module for the preradical α_N^M to be essentially idempotent.

It is possible to define the dual concept as, a preradical σ is essentially coidempotent if $\bar{\sigma}(M) = M$ then $\sigma(M) = M$ for any left R -module M . All the previous results in their dual versions are preserved.

Let σ a preradical over R -Mod. If $\widehat{\sigma} = 0$ then σ is called a strongly nilpotent preradical. The class of all strongly nilpotent preradicals is denoted by R -stn. It is remarked that R -stn is closed under infimum. Easily all atoms are idempotents or strongly nilpotents. Also the class R -stn is closed under subpreradicals, that is, let σ and τ be preradicals over R -Mod such that $\sigma \leq \tau$ and $\tau \in R$ -stn then $\sigma \in R$ -stn.

4. PREHEREDITARY PRERADICALS

Let σ be a preradical over R -Mod. Then σ is called a prehereditary preradical if the class \mathbb{T}_σ is hereditary. It is observed that σ is prehereditary if and only if $\widehat{\sigma}$ is hereditary, since $\mathbb{T}_\sigma = \mathbb{T}_{\widehat{\sigma}}$. It is remembered that a preradical σ is hereditary and only if σ is idempotent and \mathbb{T}_σ is hereditary. This means that prehereditary preradicals are a generalization of hereditary preradicals, the requirement of being idempotent is discarded. As for any family of preradicals $\{\sigma_i\}_{i \in I}$, it is true that $\mathbb{T}_{\bigwedge_{i \in I} \sigma_i} = \bigcap_{i \in I} \mathbb{T}_{\sigma_i}$, and the infimum of hereditary pretorsion classes is an hereditary pretorsion class. Then it follows:

Proposition 7. *Let $\{\sigma_i\}_{i \in I}$ a family of prehereditary preradicals over R -Mod. Then $\bigwedge_{i \in I} \sigma_i$ is a prehereditary preradical.*

Let σ be a preradical over R -Mod. It is possible to construct the least prehereditary preradical over σ . It will be denoted by σ^\square , and it results the infimum of all prehereditary preradicals over σ . The class of all prehereditary preradicals is denoted by R -pher, and it follows that R -lep \subseteq R -pher. This implies that $\sigma^\square \leq \widetilde{\sigma}$ for any preradical over R -Mod σ . It is observed that a preradical σ is prehereditary if and only if $\sigma^\square = \sigma$. The next proposition tells that the operator \square over R -pr is a closure operator.

Proposition 8. *The assignation $\square : R$ -pr \longrightarrow R -pr given by $\sigma \mapsto \sigma^\square$ for any preradical over R -Mod is a monotone, inflatory and idempotent operator over R -pr.*

Remark 5. *Let σ be a preradical over R -Mod. Then $\widetilde{\sigma^\square} = \widetilde{\sigma}$.*

For a left max ring R (a ring is left max if every no zero left module has a maximal submodule), it is considered the Jacobson radical J . Then, in general, J is not idempotent and $\mathbb{T}_J = \{0\}$, which means that J is a prehereditary preradical. In particular for any prime p and positive integer n , \mathbb{Z}_{p^n} is a max ring and the Jacobson radical is $\omega_0^{p^{n-1}}\mathbb{Z}_{p^n}$. Moreover, $\omega_0^{p^k}\mathbb{Z}_{p^n}$ with $k = 1, \dots, n-1$ is a prehereditary preradical which is not an hereditary preradical.

Proposition 9. *Let J be the Jacobson radical. It is equivalent for J :*

- (1) J is a prehereditary preradical.
- (2) $\mathbb{T}_J = \{0\}$.
- (3) $\widehat{J} = 0$.
- (4) R is a left max ring.

Proof. The interesting part is (1) implies (2) and the others are quite obvious. Let M be a no zero left R -module and let $x \in M$ a no zero element, as Rx has a maximal left submodule $J(Rx) \neq Rx$ then $J(M) \neq M$. ■

Lemma 1. *Let σ be a preradical over $R\text{-Mod}$. Then $\sigma \leq \widehat{\sigma^\square}$ if and only if $\widetilde{\sigma} = \sigma^\square$.*

Proof. If $\sigma \leq \widehat{\sigma^\square}$ and as $\widehat{\sigma^\square}$ is a left exact preradical, then $\widetilde{\sigma} \leq \widehat{\sigma^\square} \leq \sigma^\square$ which implies $\widetilde{\sigma} = \sigma^\square$. In the case that $\widetilde{\sigma} = \sigma^\square$, it follows that $\sigma \leq \sigma^\square = \widetilde{\sigma} = \widehat{\widetilde{\sigma}} = \widehat{\sigma^\square}$. ■

The next proposition says that the operator \square preserves the idempotency.

Proposition 10. *Let σ be a preradical over $R\text{-Mod}$. If σ is idempotent then σ^\square is left exact.*

Proof. As $\sigma \leq \sigma^\square$ then $\sigma \leq \widehat{\sigma} \leq \widehat{\sigma^\square}$ and by the previous proposition the result is followed. ■

For any preradical σ over $R\text{-Mod}$ it is assigned the set of left ideals $\mathbb{I}_\sigma = \{ {}_R I \leq R \mid R/I \in \mathbb{T}_\sigma \}$. This set is always closed under over ideals, which means, that if $I \in \mathbb{I}_\sigma$ and ${}_R J \leq R$ is such that $I \subseteq J$ then $J \in \mathbb{I}_\sigma$, this is because \mathbb{T}_σ is closed under quotients. The next proposition give sufficient and necessary conditions for the set \mathbb{I}_σ to be a linear filter.

Proposition 11. *Let σ be a preradical over $R\text{-Mod}$. Then σ is prehereditary if and only if \mathbb{I}_σ is a linear filter.*

Proof. If σ is prehereditary, then the proof that \mathbb{I}_σ is a linear filter is the same proof as when σ is left exact. Now, let τ be the left exact preradical induced by \mathbb{I}_σ and let M be a τ -torsion left R -module. Then $\text{ann}(x) \in \mathbb{I}_\sigma$ for any $x \in M$. Therefore $\sigma(Rx) = Rx$ for any $x \in M$, from this $\sigma(M) = M$ and $\mathbb{T}_\tau \subseteq \mathbb{T}_\sigma$. Let M a left R -module which is not τ -torsion. Then there is $x \in M$ with $\tau(Rx) \neq Rx$. As $Rx \cong R/\text{ann}(x)$ and $\mathbb{I}_\sigma = \mathbb{I}_\tau$ it follows that $\sigma(Rx) \neq Rx$ and $\sigma(M) \neq M$. Hence $\mathbb{T}_\tau \subseteq \mathbb{T}_\sigma$. ■

Corollary 3. *Let σ be a preradical over $R\text{-Mod}$. Then \mathbb{T}_σ is an hereditary torsion class if and only if \mathbb{I}_σ is a Gabriel filter.*

Proof. From the previous result \mathbb{I}_σ is a linear filter. The fact that \mathbb{T}_σ is an hereditary torsion class implies that \mathbb{I}_σ is a Gabriel filter. As in the previous proof, $\mathbb{T}_\tau \subseteq \mathbb{T}_\sigma$ where τ is the left exact radical induced by \mathbb{I}_σ . ■

Let σ be a preradical over $R\text{-Mod}$ it is called costable if \mathbb{F}_σ is closed under injective hulls.

Proposition 12. *Let σ be a radical over $R\text{-Mod}$. If σ is costable then σ is left exact.*

Proof. See [1]. ■

Proposition 13. *Let σ be an essentially idempotent prehereditary preradical over $R\text{-Mod}$. Then σ is costable.*

Proof. Let M be a left R -module such that $\sigma(M) = 0$. Then $\widehat{\sigma}(M) = 0$, since $\widehat{\sigma}(M) = \widehat{E(M)} \cap M$ and $M \leq E(M)$. It follows that $\widehat{\sigma}(E(M)) = 0$, as σ is essentially idempotent $\sigma(EM) = 0$. ■

Proposition 14. *Let $\{\sigma_i\}_{i \in I}$ be a family of prehereditary radicals over $R\text{-Mod}$. Then $\bigwedge_{i \in I} \sigma_i$ is a prehereditary radical.*

Proposition 15. *Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module. If $M \in \mathbb{T}_\sigma$ then $M \subseteq \sigma^\square(N)$ for any left R -module N such that $M \leq N$, in particular, $M \subseteq \sigma^\square(E(M))$.*

A counterexample that the inverse proposition is not valid, let p and q be different primes and it is defined $\sigma = \alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p^\infty} \vee \alpha_{\mathbb{Z}_q}^{\mathbb{Z}_q^\infty}$. As σ is prehereditary (since $\widehat{\sigma} = 0$) it follows that $\sigma^\square = \sigma$. Also $\sigma(\mathbb{Z}_q^\infty) = \mathbb{Z}_q$ then $\mathbb{Z}_q \subseteq \sigma(E(\mathbb{Z}_q))$, but $\sigma(\mathbb{Z}_q) = 0$.

Proposition 16. *Let S be a simple left R -module. Then $\alpha_S^S(E(M)) = \alpha_S^S(M)$ for any left R -module M .*

Corollary 4. *Let σ be an atom in $R\text{-pr}$. Then σ is prehereditary.*

Proof. As σ is an atom it must be of the form $\alpha_S^{E(S)}$ for some simple left R -module S . Since σ is an atom $\widehat{\sigma}$ must be 0 or σ . In the first case σ is prehereditary and in the second case σ is idempotent which means that S is injective. So by the previous proposition $\sigma(M) = \alpha_S^{E(S)}(M) = \alpha_S^S(M) = \alpha_S^S(E(M)) \cap M = \sigma(E(M)) \cap M$ which means that σ is left exact, therefore prehereditary. ■

Lemma 2. *Let M be a left R -module and let N be a fully invariant submodule of M . If M is L -injective for all $L \in \mathbb{T}_{\omega_N^M}$ then ω_N^M is a prehereditary preradical.*

Proof. Let K be a ω_N^M -torsion module, let L be a submodule of K and let $f : L \rightarrow M$ be an R -morphism. So there is $g : K \rightarrow M$ such that $g|_L = f$ which means that $f^{-1}(N) = g^{-1}(N) \cap L = K \cap L = L$ implying that L is ω_N^M -torsion. ■

Proposition 17. *Let σ be a left exact preradical over $R\text{-Mod}$ and let τ be a strongly nilpotent preradical over $R\text{-Mod}$. If $\sigma \wedge \tau = 0$ then $\sigma \vee \tau$ is a prehereditary preradical.*

Proof. It is noticed that $\widehat{\sigma \vee \tau} = \widehat{\sigma} \vee \widehat{\tau} = \sigma$. ■

The last result tells how to construct a infinite family of non trivial prehereditary preradicals (understanding by non trivial as no left exact). Let p and q be different primes. Then by the previous proposition $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p} \vee \alpha_{\mathbb{Z}_q}^{\mathbb{Z}_q^\infty}$ is a prehereditary preradical which is not idempotent (meaning that is not left exact), neither is radical, and its torsion class is non trivial.

5. ESSENTIALITY WITH RESPECT TO A PRERADICAL

Proposition 18. *Let M be a non singular left R -module and let N a submodule of M . Then $N \trianglelefteq M$ if and only if M/N is singular.*

Proposition 19. *Let M be a left R -module and let N and K be submodules of M . Then:*

- (1) *Let $x \in M$. If $N \trianglelefteq M$ then $(N : x) \trianglelefteq R$.*
- (2) *If $K \trianglelefteq M$ and $L \trianglelefteq M$ then $N \cap K \trianglelefteq M$.*
- (3) *If $K \trianglelefteq N$ and $N \trianglelefteq M$ then $K \trianglelefteq M$.*
- (4) *If $K \trianglelefteq M$ and $K \leq N$ then $K \trianglelefteq N$ and $N \trianglelefteq M$.*

With the previous two propositions it is possible to think in a kind of essentiality respect to a preradical σ . Let M be a left R -module and let N be a submodule of M , it is said that N is σ -dense in M if $\sigma(M/N) = M/N$. This fact is denoted by $N \trianglelefteq_\sigma M$. It may be thought as σ -essentiality. In fact, if σ is prehereditary most of the last properties are preserved.

Proposition 20. *Let M be a left R -module, let N and K be submodules of M and let σ be a preradical over $R\text{-Mod}$. Then:*

- (1) *If $K \trianglelefteq_\sigma M$ and $K \leq N$ then $N \trianglelefteq_\sigma M$.*

When σ is prehereditary,

- (2) *Let $x \in M$. If $N \trianglelefteq_\sigma M$ then $(N : x) \trianglelefteq_\sigma R$.*
 (3) *If $K \trianglelefteq_\sigma M$ and $L \trianglelefteq_\sigma M$ then $N \cap K \trianglelefteq_\sigma M$.*
 (4) *If $K \trianglelefteq_\sigma M$ and $K \leq N$ then $K \trianglelefteq_\sigma N$.*
 (5) *If $N \leq M$ and $K \trianglelefteq_\sigma M$ then $K \cap N \trianglelefteq_\sigma N$.*

When σ is essentially coidempotent,

- (6) *If $K \trianglelefteq_\sigma N$ and $N \trianglelefteq_\sigma M$ then $K \trianglelefteq_\sigma M$.*

Proof.

- (1) Follows from the fact that \mathbb{T}_σ is closed under quotients.
 (2) It is considered the next equalities $R/(N : x) = R/\text{ann}(x+N) \cong R(x+N) \leq M/N$.
 (3) Let $\pi : M \longrightarrow M/N \times M/K$ be the morphism induced by the canonical projections, as $M/N \times M/K$ is a σ -torsion left R -module, $\ker \pi = N \cap K$ and $M/(N \cap K)$ is isomorphic to a submodule of $M/N \times M/K$ then $M/(N \cap K)$ is a σ -torsion left R -module.
 (4) Follows from the fact that \mathbb{T}_σ is closed under submodules.
 (5) Follows by the second isomorphism theorem and the fact that \mathbb{T}_σ is closed under submodules.
 (6) Follows from the fact that \mathbb{T}_σ is closed under extensions. ■

Proposition 21. *Let σ and τ be preradicals over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . If $\sigma \leq \tau$ and $N \trianglelefteq_\sigma M$ then $N \trianglelefteq_\tau M$.*

Proof. Follows from the fact that $\mathbb{T}_\sigma \subseteq \mathbb{T}_\tau$. ■

Proposition 22. *Let σ be a preradical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . Then $N \trianglelefteq_\sigma M$ if and only if $N \trianglelefteq_{\hat{\sigma}} M$.*

Proof. Follows from the fact that $\mathbb{T}_\sigma = \mathbb{T}_{\hat{\sigma}}$. ■

Proposition 23. *Let σ be a prehereditary preradical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . If M is σ -torsionfree and $N \trianglelefteq_\sigma M$ then $N \trianglelefteq M$.*

Proof. Let $x \in M$, then by the second isomorphism theorem $Rx/(Rx \cap N) \cong (Rx + N)/N$ which implies that $Rx + N \trianglelefteq_\sigma M$. Also $Rx/(Rx \cap N)$ is of σ -torsion, since Rx is σ -torsionfree. If $Rx \cap N = 0$ then Rx is σ -torsion. This implies $Rx = 0$ and $x = 0$. Which means that if $x \neq 0$ then $Rx \cap N \neq 0$. ■

6. PURE SUBMODULES RESPECT TO A PRERADICAL

Let σ be a preradical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . It is said that N is σ -pure submodule of M if M/N is σ -torsionfree.

Proposition 24. *Let σ be a preradical over $R\text{-Mod}$, let M be a left R -module and let $\{M_i\}_{i \in I}$ be a family of σ -pure submodules of M . Then $\bigcap_{i \in I} M_i$ is a σ -pure submodule of M .*

Proof. Let $\pi : M \longrightarrow \prod_{i \in I} M/M_i$ be the morphism induced by canonical projections, since $\prod_{i \in I} M/M_i$ is σ -torsionfree it follows that $M/\bigcap_{i \in I} M_i$ is σ -torsionfree. ■

For a submodule N of a left R -module M , it should be considered the least σ -pure submodule of M that contains N . It is denoted by N_σ^M . It is described by

$$N_\sigma^M = \bigcap \{K \leq M \mid N \leq K, M/K \in \mathbb{F}_\sigma\}$$

As the last proposition states it is σ -pure in M and contains N . The submodule N_σ^M is called the σ -purification of N in M .

Proposition 25. *Let σ be a preradical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . Then $\bar{\sigma}(M/N) = N_\sigma^M/N$.*

Corollary 5. *Let σ be a preradical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . Then $N_\sigma^M = N_{\bar{\sigma}}^M$.*

Proposition 26. *Let σ be a preradical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . Then N is σ -pure in M if and only if $N = N_\sigma^M$.*

Proof. As $\sigma(M/N) = 0$ implies $\bar{\sigma}(M/N) = 0$ the result follows. ■

Remark 6. *Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module. Then $\bar{\sigma}(M) = 0_\sigma^M$.*

Lemma 3. *Let σ be a preradical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . If N is σ -pure in M and $N \in \mathbb{T}_\sigma$ then $\sigma(M) = N$.*

Proof. First as N is σ -pure in M then $\sigma(M/N) = 0$. This way

$$\begin{aligned} \sigma(M) &\subseteq \bar{\sigma}(M) \\ &= \bigcap \{K \leq M \mid \sigma(M/K) = 0\} \subseteq N \end{aligned}$$

By other side $N \leq M$ implies $N = \sigma(N) \leq \sigma(M)$. ■

Remark 7. *Let σ be an idempotent radical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . Then N_σ^M/N is a σ -torsion module.*

7. INJECTIVITY RESPECT TO A PRERADICAL

Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module. It is said that M is σ -injective if $f : K \rightarrow M$ is a R -morphism and $K \trianglelefteq_\sigma N$ then there is a morphism $g : N \rightarrow M$ with $g|_K = f$. This concept is a generalization of injectivity respect an hereditary torsion theory, as a reference is the book (2). The first thing that is observed is that: if M is also a σ -torsion module then M is quasi-injective.

Proposition 27. *Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module. Then (1) and (2) are equivalent and imply (3), (3) implies (4) and (5), (4) implies (6) and (5) implies (6). If σ is an idempotent radical then (1), (2), (3) and (5) are equivalent. If σ is prehereditary then (5) and (6) are equivalent. If σ is a left exact radical then all are equivalent.*

(1) M is σ -pure in $E(M)$

(2) If M is a submodule of a left R -module N , then there exist a σ -pure submodule K of N that contains M and M is a direct summand of K .

- (3) $\text{Ext}_R^1(N, M) = 0$ for any σ -torsion left R -module N .
- (4) $\text{Ext}_R^1(R/I, M) = 0$ for any $I \in \mathbb{I}_\sigma$.
- (5) M is σ -injective
- (6) M is σ -injective respect to R

Proof. (1) \Rightarrow (2) If $M \leq N$ then $E(N) = E(M) \oplus L$. It is put $L' = N \cap L$ then $N/(M \oplus L')$ is isomorphic to a submodule of $E(N)/(M \oplus L)$. Since it is considered the morphism $f : N \rightarrow E(N)/(M \oplus L)$ with $f = gh$ where $g : E(N) \rightarrow E(N)/(M \oplus L)$ is the canonical projection and $h : N \rightarrow E(N)$ is the canonical inclusion. So $\ker f = N \cap (M \oplus L) = M \oplus L'$. By other side $E(N)/(M \oplus L) \cong (E(M) \oplus L)/(M \oplus L) \cong E(M)/M$ which by hypothesis is σ -torsionfree. That is why $N/(M \oplus L')$ is σ -torsionfree and $K = M \oplus L'$ is σ -pure in N .

(2) \Rightarrow (1) As $M \leq E(M)$, by hypothesis then there is $K \leq E(M)$ with $M \oplus K$ σ -pure in $E(M)$. But $M \leq E(M)$ which means that $K = 0$. Therefore M is σ -pure in $E(M)$.

(1) \Rightarrow (3) It is considered the short exact sequence

$$0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$$

and it is obtained a exact sequence

$$\text{Hom}_R(N, E(M)/M) \rightarrow \text{Ext}_R^1(N, M) \rightarrow \text{Ext}_R^1(N, E(M))$$

where N is a σ -torsion left R -module. As $E(M)/M$ is σ -torsionfree, this implies $\text{Hom}_R(N, E(M)/M) = 0$. Also $E(M)$ is injective, so $\text{Ext}_R^1(N, E(M)) = 0$. From this follows that $\text{Ext}_R^1(N, M) = 0$.

(3) \Rightarrow (5) It is taken a short exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N/N' \rightarrow 0$ such that N/N' is a σ -torsion module. It is induced the following short exact sequence

$$0 \rightarrow \text{Hom}_R(N/N', M) \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Ext}_R^1(N/N', M)$$

and by hypothesis the last module is zero.

(2) \Rightarrow (4), (4) \Rightarrow (6) and (5) \Rightarrow (6) are obvious.

(6) \Rightarrow (5) In the same way as the proof of the Baer's criterion.

(5) \Rightarrow (1) As $M \leq_{\sigma} M_{\sigma}^{E(M)}$ since σ is an idempotent radical, there is an R -morphism $\alpha : M_{\sigma}^{E(M)} \rightarrow M$ such that $\alpha|_M = 1_M$. Which implies that α is an epimorphism. It is noticed that $\ker \alpha \cap M = \ker 1_M = 0$ since $M \leq M_{\sigma}^{E(M)}$. It follows that α is a monomorphism, so $M = M_{\sigma}^{E(M)}$. ■

By the last proposition it is observed that it is sufficient to ask to a preradical to be an idempotent radical to speak about relative injectivity. The only thing that may not be assured is the Baer's criterion. In the other hand it is sufficient to ask a preradical to be prehereditary to have Baer's criterion.

Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module, it is defined the σ -injective hull of M as $M_{\sigma}^{E(M)}$ and it is denoted by $E_{\sigma}(M)$.

Remark 8. *Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module. Then:*

- (1) $E_{\sigma}(M)$ is σ -injective
- (2) $M \leq E_{\sigma}(M)$
- (3) If σ is an idempotent radical then $M \leq_{\sigma} E_{\sigma}(M)$.

This three properties characterizes the σ -injective hull as the next proposition tells. It is an analogous characterization of the usual injective hull as an injective essential extension, but now it is asked to be σ -dense extension. The hypothesis over σ is to be an idempotent radical.

Proposition 28. *Let σ be a preradical over $R\text{-Mod}$, let K be a σ -injective left R -module and let M be a σ -dense dense submodule of K . If σ is an idempotent radical then $K = E_{\sigma}(M)$.*

Proof. As M is essential in K without loss of generality it may reduced to the case when $K \leq E(M)$. So $E(K) = E(M)$ and K is σ -pure in $E(M)$ then by lemma 3 the result is followed. ■

Remark 9. *Let σ be an idempotent radical over $R\text{-Mod}$ and let M be a left R -module. Then M is σ -injective if and only if $E_{\sigma}(M) = M$.*

Proposition 29. *Let σ be an idempotent radical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . If M is σ -injective and N is σ -pure in M then N is σ -injective.*

Proof. Let K be a σ -torsion left R -module. It is considered the next short exact sequence:

$$\operatorname{Hom}_R(K, M/N) \longrightarrow \operatorname{Ext}_R^1(K, N) \longrightarrow \operatorname{Ext}_R^1(K, M)$$

As $\operatorname{Ext}_R^1(K, M) = 0$ and $\operatorname{Hom}_R(K, M/N) = 0$ since M is σ -injective and M/N is σ -torsion free. It follows that $\operatorname{Ext}_R^1(K, N) = 0$, therefore N is σ -injective. ■

Remark 10. Let σ be an idempotent radical over $R\text{-Mod}$ and let M be a left R -module. If M is σ -injective σ -torsion module then $\sigma(E(M)) = M$.

Remark 11. Let σ be a radical over $R\text{-Mod}$ and let M be a left R -module. If $\sigma(E(M)) = M$ then M is σ -injective.

Let M a left R -module. It is defined $\Omega(M)$ as the set of all left ideals that contain $\operatorname{ann}(x)$ for some $x \in M$.

Lemma 4 (Technical). Let M be a left R -module. Then M is quasiinjective if and only if for any left ideal L and for any R -morphism $\alpha : L \longrightarrow M$ with $\ker \alpha \in \Omega(M)$ there is an R -morphism $\beta : R \longrightarrow M$ such that $\beta|_L = \alpha$.

Proof. [3, lemma 2]. ■

The previous lemma is used in the proof of the following proposition.

Proposition 30. Let σ be a preradical over $R\text{-Mod}$ and let $\{M_i\}_{i \in I}$ be a family of left R -modules. Then $\prod_{i \in I} M_i$ is σ -injective if and only if M_i is σ -injective for any $i \in I$.

Proposition 31. Let σ be a preradical over $R\text{-Mod}$ and let M be a σ -torsion left R -module. If σ is an idempotent radical the (1) implies (2), if σ is prehereditary then (2) implies (1) and if σ is a left exact radical (1) and (2) are equivalent.

(1) M is σ -injective

(2) (a) M is quasi-injective.

(b) If $I \in \mathbb{I}_\sigma$ and I' is a left ideal such that $I' \subseteq I$ and I/I' can be embedded in M then $I' = I \cap \operatorname{ann}(x)$ for some $x \in M$.

Proof. The arguments of the proposition (4.2) of (4). ■

Proposition 32. *Let M be a quasi-injective left R -module. If $\omega_M^{E(M)}$ is a radical then M is σ -injective for any preradical σ such that $\sigma(E(M)) = M$.*

Proof. If $\omega_M^{E(M)}$ is a radical, as $\sigma(E(M)) = M$, this implies $\sigma \leq \omega_M^{E(M)}$. Then $\sigma(E(M)/M) \leq \omega_M^{E(M)}(E(M)/M) = 0$ which means that M is σ -pure in $E(M)$. Therefore σ -injective. ■

Examples

Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module. As it is seen $E_\sigma(M)/M = \bar{\sigma}(E(M)/M)$. Let R be the ring of the integers \mathbb{Z} and it is considered $\sigma = Soc, t, d, J$ where Soc is the socle, t the torsion part, d the divisible part and J the Jacobson radical. Then $E_\sigma(\mathbb{Z}) = \mathbb{Q}$ and $E_\sigma(\mathbb{Z}_{p^k}) = \mathbb{Z}_{p^\infty}$ with p a prime number and k a natural number. But if $\sigma = \alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}$ with p a prime number then $E_\sigma(\mathbb{Z}) = \{\frac{a}{p^m} \in \mathbb{Q} \mid a, m \in \mathbb{N}\}$, $E_\sigma(\mathbb{Z}_{q^k}) = \mathbb{Z}_{q^\infty}$ when $p = q$ and $E_\sigma(\mathbb{Z}_{q^k}) = \mathbb{Z}_{q^k}$ if $p \neq q$ with q a prime number.

8. PSEUDOCOMPLEMENTED SUBMODULES RELATIVE TO A PRERADICAL

Let σ be a preradical over $R\text{-Mod}$, let M be a left R -module and let N be a submodule of M . It is said that N is σ -pseudocomplemented in M if there is a submodule K of M such that $N \cap K = 0$, $N \oplus K \leq M$ and $N \oplus K$ is σ -dense in M . The submodule K is called a σ -pseudocomplement of N in M . It is remarked that this concept is similar to the concept of μ -complemented but is not same.

Proposition 33. *Let σ be a essentially coidempotent preradical over $R\text{-Mod}$ and let M, N and K be a left R -modules with $K \leq N \leq M$. If K is σ -pseudocomplemented in N and N is σ -pseudocomplemented in M , then K is σ -pseudocomplemented in M .*

Proof. By hypothesis there are K' submodule of N and N' submodule of M such that $K \cap K' = 0$, $N \cap N' = 0$, $K \oplus K' \leq N$, $N \oplus N' \leq M$, $K \oplus K' \leq_\sigma N$ and $N \oplus N' \leq_\sigma M$. It is proposed $K' \oplus N'$ as the σ -pseudocomplement of K in M . Immediately $K \oplus K' \oplus N' \leq M$. Next it is considered the following short exact sequence:

$$0 \longrightarrow (N \oplus N')/(K \oplus K' \oplus N') \longrightarrow M/(K \oplus K' \oplus N') \longrightarrow M/(N \oplus N') \longrightarrow 0$$

As $(N \oplus N')/(K \oplus K' \oplus N') \cong N/(K \oplus K') \in \mathbb{T}_\sigma$, $M/(N \oplus N') \in \mathbb{T}_\sigma$ and \mathbb{T}_σ is closed under extensions, then $M/(K \oplus K' \oplus N') \in \mathbb{T}_\sigma$. Therefore the proposition is proved. ■

Proposition 34. *Let σ be a prehereditary preradical over $R\text{-Mod}$ and let M, N and K be a left R -modules with $K \leq N \leq M$. If K is σ -pseudocomplemented in M then K is σ -pseudocomplemented in N .*

Proof. By hypothesis there is K' submodule of M such that $K \cap K' = 0$, $K \oplus K' \leq M$ and $K \oplus K' \leq_\sigma M$. It is proposed $K'' = N \cap K'$ as the σ -pseudocomplement of K in N . First it is obvious that $K'' \cap N = 0$. Second

$$\begin{aligned} K \oplus K'' &= K \oplus (N \cap K') \\ &= N \cap (K \oplus K') \leq N \end{aligned}$$

At last, as $K \oplus K' \leq_\sigma M$, then $K \oplus K'' = N \cap (K \oplus K') \leq_\sigma N$ as it is desired. ■

Let σ be a prehereditary preradical over $R\text{-Mod}$ and let M be a left R -module. $\text{Subp}_\sigma(M)$ denotes the set of all submodules of M that are σ -pseudocomplemented.

Remark 12. *Let σ and τ be preradicals over $R\text{-Mod}$ and let N and M be left R -modules. Then:*

- (1) $M \in \text{Subp}_\sigma(M)$.
- (2) $0 \in \text{Subp}_\sigma(M)$.
- (3) If $N \leq_\sigma M$ then $N \in \text{Subp}_\sigma(M)$.
- (4) If M is a σ -torsion module then $\text{Subp}_\sigma(M) = \text{Sub}(M)$.
- (5) If N is a direct summand of M then $N \in \text{Subp}_\sigma(M)$.
- (6) If $\sigma \leq \tau$ then $\text{Subp}_\sigma(M) \subseteq \text{Subp}_\tau(M)$.

Let σ be a prehereditary preradical over $R\text{-Mod}$, \mathbb{E}_σ denotes the class of all σ -injective left modules.

Proposition 35. *Let σ and τ be preradicals over $R\text{-mod}$. If $\text{Subp}_\sigma(M) = \text{Subp}_\tau(M)$ for any left R -module M then $\mathbb{E}_\sigma = \mathbb{E}_\tau$.*

Proof. Let E be a σ -injective left R -module, let M be a left R -module, N a τ -dense submodule of M and $\alpha : N \rightarrow E$ an R -morphism. First it is observed that $N \in \text{Subp}_\tau$. Then it has a σ -pseudocomplemented N' in M . So it is considered the morphism $\alpha \oplus 0 : N \oplus N' \rightarrow E$ since $N \oplus N' \leq_\sigma M$. Then there is a morphism $\beta : M \rightarrow E$ such that $\beta|_{N \oplus N'} = \alpha \oplus 0$. So $\beta|_N = \alpha$ which proves that E is τ -injective. ■

Corollary 6. *Let Z be the singular preradical and let E be a left R -module. Then E is injective if and only if E is Z -injective.*

9. ABSOLUTE σ -PURE

Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module. It is said that M is absolutely σ -pure if M is σ -torsionfree and σ -injective.

Proposition 36. *Let σ be a preradical over $R\text{-Mod}$ and let M be a left R -module. Then (1) \Rightarrow (2) and if σ is idempotent then are equivalent.*

- (1) M is absolutely σ -pure.
- (2) For any left R -module N , for any σ -dense submodule of N , K , and for any R -morphism $\alpha : K \longrightarrow M$ there is a unique R -morphism $\beta : N \longrightarrow M$ such that $\beta|_K = \alpha$.

Proof. (1) \Rightarrow (2) Let β and β' be R -morphisms such that $\beta|_K = \alpha$ and $\beta'|_K = \alpha$. Then $K \leq \ker(\beta - \beta')$ so there is a morphism $\gamma : N/K \longrightarrow M$ given by $\gamma(x + K) = (\beta - \beta')(x)$ for any $x + K \in N/K$. As N/K is a σ -torsion module and M is a σ -torsion free module. It follows that $\gamma = 0$. Therefore $\beta = \beta'$.

(2) \Rightarrow (1) It must seen that $\sigma(M) = 0$. So 0 is σ -essential submodule of $\sigma(M)$ and there are two morphisms that extend the morphism $0 : 0 \longrightarrow M$, the inclusion $i : \sigma(M) \longrightarrow M$ and $0 : \sigma(M) \longrightarrow M$. By the uniqueness $i = 0$. It follows that $\sigma(M) = 0$. ■

Proposition 37. *Let σ be a preradical costable over $R\text{-Mod}$ and let M be a left R -module. If M is σ -torsion free and M is σ -pure in any σ -torsion free module that contains it then M is absolutely σ -pure.*

Proof. As M is σ -torsion free the $E(M)$ is σ -torsion free. So M is σ -pure in $E(M)$ which implies that M is σ -injective. ■

Proposition 38. *Let σ be an essentially idempotent preradical over $R\text{-Mod}$ and let M be a left R -module. If M is absolutely σ -pure then M is σ -torsion free and M is σ -pure in any σ -torsion free module that contains it.*

Proof. Let M' be a σ -torsion free R -module that contains M . Then there is a submodule of M' , N , such that $M \oplus N$ is σ -pure in M' . So it is observed the following short exact sequence:

$$0 \longrightarrow (M \oplus N)/M \longrightarrow M'/M \longrightarrow M'/(M \oplus N) \longrightarrow 0$$

Now, as $(M \oplus N)/M \cong N$ which is σ -torsion free and $M'/(M \oplus N)$ is σ -torsion free, then M'/M is σ -torsion free. ■

10. AUTOCOSTABLE PRERADICALS

Let σ be a preradical over $R\text{-Mod}$. It is said the σ is autocostable if \mathbb{F}_σ is closed under σ -injective hulls.

Remark 13. Let σ be a preradical over $R\text{-Mod}$. If σ is costable then it is autocostable.

Proposition 39. Let σ be a preradical over $R\text{-Mod}$. If σ is an autocostable essentially idempotent preradical then σ is a costable preradical.

Proof. Let M be a σ -torsion free left R -module. It is considered the following exact sequence:

$$0 \longrightarrow E_\sigma(M) \longrightarrow E(M) \longrightarrow E(M)/E_\sigma(M) \longrightarrow 0$$

as $E(M)/E_\sigma(M)$ is σ -torsion free. Then $E(M)$ is σ -torsion free. ■

Corollary 7. Let σ be a preradical over $R\text{-Mod}$. If σ is an autocostable essentially idempotent radical then σ is left exact radical.

11. LOCALIZATION

Let σ be a preradical over $R\text{-Mod}$. It is defined an assignation Q_σ from $R\text{-Mod}$ to $R\text{-Mod}$ as $Q_\sigma(M) = E_\sigma(M/\sigma(M))$ for any left R -module M . Also it is defined $\eta_M^\sigma : M \longrightarrow Q_\sigma(M)$ as the canonical projection composed with the canonical inclusion. It is observed that if σ is a left exact radical then $Q_\sigma(M)$ is an absolutely σ -pure module for any left R -module M . If $\alpha : M \longrightarrow N$ is an R -morphism it induces an R -morphism $\bar{\alpha} : M/\sigma(M) \longrightarrow N/\sigma(N)$ so it is composed with the inclusion of $N/\sigma(N)$ in $Q_\sigma(N)$ and as $Q_\sigma(N)$ is absolute σ -pure then there is a unique R -morphism $\gamma : Q_\sigma(M) \longrightarrow Q_\sigma(N)$ such that extends the composition metioned. If it is put $Q_\sigma(f) = \gamma$, it is straigh to check that in this case this assignment makes to Q_σ an endofunctor over $R\text{-Mod}$. The endofunctor is called the localization respect σ and has been studied a lot, as references are (4), (5) and (9).

Proposition 40. *Let σ be a left exact radical over $R\text{-Mod}$. Then Q_σ is idempotent and left exact.*

Proposition 41. *Let σ be a left exact radical over $R\text{-Mod}$. Then $\eta^\sigma : 1_{R\text{-Mod}} \longrightarrow Q_\sigma$ is a natural transformation.*

Proposition 42. *Let σ be a left exact radical over $R\text{-Mod}$ and let M be a left R -module. Then $\ker \eta_M^\sigma$ is a σ -torsion module and $\text{coker} \eta_M^\sigma$ is a σ -torsion free module.*

Proposition 43. *Let σ be a left exact radical over $R\text{-Mod}$. Then $\eta^\sigma \circ Q_\sigma = Q_\sigma \circ \eta^\sigma$.*

Proof. Let M be a left R -module. It is easy to verify that $\eta_{Q_\sigma(M)}^\sigma = 1_{Q_\sigma(M)}$ and $Q_\sigma(\eta_M^\sigma) = 1_{Q_\sigma(M)}$. ■

Proposition 44. *Let σ be a preradical over $R\text{-Mod}$. Then σ is an idempotent preradical if and only if $Q_\sigma \circ \sigma = 0$*

Proof. Let M be a left R -module then $(Q_\sigma \circ \sigma)(M) = E_\sigma(\sigma(M)/\sigma^2(M))$. ■

Proposition 45. *Let σ be a preradical over $R\text{-Mod}$. If $Q_\sigma \circ \sigma = \sigma \circ Q_\sigma$ then σ is an idempotent autocoestable radical.*

Proof. Is easy to see that if σ is idempotent, then by the previous proposition $\sigma \circ Q_\sigma = 0$. Which implies that $\sigma(E_\sigma(M/\sigma(M))) = 0$ for any left R -module M . So $\sigma(M/\sigma(M)) = 0$, which means that σ is a radical. This implies $\mathbb{F}_\sigma = \{M/\sigma(M) \mid M \in R\text{-Mod}\}$. Therefore the class \mathbb{F}_σ is closed under σ -injective hulls. ■

Corollary 8. *Let σ be a preradical over $R\text{-Mod}$. Then $Q_\sigma \circ \sigma = \sigma \circ Q_\sigma$ if and only if σ is a left exact radical.*

Proof. All idempotent autocostable radicals are left exact radicals. ■

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